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EXISTENCE OF A UNIQUE MAXIMAL SUBCOALGEBRA WHOSE ACTION IS INNER

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ABSTRACT

Let C be a coalgebra over a field k. Fix an algebra map $\alpha : R \to A$. Introducing the notion of cleft forms, we show that, for any measuring $\phi : C \to \mathrm{Hom}(R, A)$, there is a unique maximal subcoalgebra D of C such that $\phi|_D$ is inner.

Introduction

We work over a field k. Let C be a coalgebra, and R, A algebras (over k). A linear map $\phi : C \to \mathrm{Hom}(R, A)$ is called a *measuring*, if the action represented by ϕ

$$C \otimes R \to A, \quad c \otimes x \mapsto c[x] \qquad (= \phi(c)(x))$$

measures R to A [S, p. 139], i.e.,

$$c[1] = \varepsilon(c)1, \quad c[xy] = \sum_{(c)} c_{(1)}[x]c_{(2)}[y]$$

for $c \in C$, $x, y \in R$. If an algebra map $\alpha : R \to A$ is fixed, the notion of inner measurings is defined as follows: A measuring $\phi : C \to \mathrm{Hom}(R, A)$ is said to be *inner* (with respect to α), if there is a $*$-invertible linear map $u : C \to A$ such that $\phi = \mathrm{inn}\, u$, where $\mathrm{inn}\, u$ is determined by

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$$\operatorname{inn} u(c)(x) = \sum_{(c)} u(c_{(1)})\alpha(x)u^{-1}(c_{(2)})$$

for $c \in C$, $x \in R$. The purpose of this paper is to prove:

THEOREM 9. *For any measuring $\phi : C \to \operatorname{Hom}(R, A)$, there is a unique maximal subcoalgebra D of C such that $\phi|_D$ is inner.*

This is an affirmative answer to the coalgebra version of the following question raised by Colin Sutherland:

QUESTION [Mo, 6.3]. Let H be a Hopf algebra and let A be a left H-module algebra represented by $\phi : H \to \operatorname{End} A$. Then, is there a unique maximal Hopf subalgebra K of H such that $\phi|_K$ is inner?

To prove our Theorem 9, we introduce the notion of (*cleft*) *forms*, a modification of *Galois subalgebras* due to Doi and Takeuchi. We will show in Proposition 8 that there is a 1–1 correspondence between the cleft forms $\subset C \otimes A$ and the inner measurings $C \to \operatorname{Hom}(R, A)$. Considering cleft forms instead of inner measurings, we can argue comodule-theoretically. This method is useful for us to examine inner actions of coalgebras or Hopf algebras, as is shown in [Ma2, Section 3], too.

We write $\otimes = \otimes_k$, $\operatorname{Hom} = \operatorname{Hom}_k$ and $\operatorname{End} = \operatorname{End}_k$. Modules mean *right* modules and comodules mean *left* comodules.

Let A be an algebra and let C be a coalgebra with the structure Δ, ε. We write as usual

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}, \qquad c \in C.$$

$\operatorname{Hom}(C, A)$ is an algebra with the $*$-product [S, p. 69]. $\operatorname{Reg}(C, A)$ denotes the group of $*$-invertible linear maps $C \to A$.

$C \otimes A$, being naturally a (right) A-module, has a C-comodule structure

$$C \otimes A \xrightarrow{\Delta \otimes \mathrm{id}} C \otimes C \otimes A,$$

which is an A-module map.

DEFINITION and LEMMA 1. (a) *For $u \in \operatorname{Hom}(C, A)$, define $\hat{u} \in \operatorname{End}(C \otimes A)$ by*

$$\hat{u}(c \otimes a) = \sum_{(c)} c_{(1)} \otimes u(c_{(2)})a$$

for $c\in C$, $a\in A$. $u\mapsto \hat{u}$ *gives a* 1–1 *correspondence between* $\mathrm{Hom}(C,A)$ *and the set of C-comodule and A-module endomorphisms of* $C\otimes A$.

(b) *Let* $u, v\in\mathrm{Hom}(C,A)$. *We have*

$$\widehat{u*v}=\hat{u}\circ\hat{v},$$

$$\hat{\varepsilon}=\mathrm{id}_{C\otimes A}$$

so that u is $*$*-invertible* $\Leftrightarrow$ $\hat{u}$ *is an automorphism*.

PROOF. (a) It follows from [D, Prop. 3, p. 33] that the above set is identified with

$$\mathrm{Hom}_{-A}(C\otimes A,A)=\mathrm{Hom}(C,A)$$

via $\hat{u}\leftrightarrow u$.

(b) This is verified easily. Q.E.D.

Let B be a subalgebra of A.

DEFINITION 2. Let $N\subset C\otimes A$ be a C-subcomodule as well as a (right) B-submodule. N is called a *B-form* of $C\otimes A$, if the canonical map

$$N\otimes_B A\to C\otimes A,\quad n\otimes a\mapsto na$$

is an isomorphism. A B-form $N\subset C\otimes A$ is said to be *cleft*, if there is a (left) C-comodule and (right) B-module isomorphism $N\simeq C\otimes B$.

Here the notion of B-forms is a modification of *Galois subalgebras* due to Doi and Takeuchi [DT, Def. 6.8, pp. 509–510].

DEFINITION 3. For $u\in\mathrm{Reg}(C,A)$, denote by

$$N(u)$$

the image of $\hat{u}\,|_{C\otimes B}: C\otimes B\to C\otimes A$.

LEMMA 4. *Let* $u, v\in\mathrm{Reg}(C,A)$.

(a) $N(u)$ *is a cleft B-form of* $C\otimes A$. *Conversely, any cleft B-form of* $C\otimes A$ *is of the form* $N(u)$ *for some* $u\in\mathrm{Reg}(C,A)$.

(b) $N(u)\supset N(v)\Leftrightarrow u^{-1}*v\in\mathrm{Hom}(C,B)$.

(c) $N(u)=N(v)\Leftrightarrow u^{-1}*v\in\mathrm{Reg}(C,B)$.

PROOF. (a) Since $u\in\mathrm{Reg}(C,A)$, $\hat{u}\,|_{C\otimes B}: C\otimes B\to N(u)$ is an isomorphism. So $N(u)$ will be a cleft B-form of $C\otimes A$ provided the canonical map $N(u)\otimes_B A \to C\otimes A$ is an isomorphism. This follows by viewing the commutative diagram:

$$\begin{array}{ccc} C\otimes A & \underset{\hat{u}}{\overset{\sim}{\to}} & C\otimes A \\ \| & & \uparrow \text{cano.} \\ (C\otimes B)\otimes_B A & \underset{\hat{u}|_{C\otimes B}\otimes \mathrm{id}}{\overset{\sim}{\longrightarrow}} & N(u)\otimes_B A \end{array}$$

Conversely, suppose N is a cleft B-form $\subset C\otimes A$ and let $f: C\otimes B \overset{\sim}{\to} N$ be a C-comodule and B-module isomorphism. By Lemma 1, there is $u\in \mathrm{Reg}(C,A)$ satisfying:

$$\begin{array}{ccccc} C\otimes A & \underset{\hat{u}}{\overset{\sim}{\to}} & C\otimes A & & \\ \| & \circlearrowleft & \wr\uparrow \text{cano.} & & \\ (C\otimes B)\otimes_B A & \underset{f\otimes \mathrm{id}}{\overset{\sim}{\to}} & N\otimes_B A & \circlearrowright & \text{incl.} \\ \uparrow & \circlearrowleft & \uparrow & & \\ C\otimes B & \underset{f}{\overset{\sim}{\to}} & N & & \end{array}$$

Hence $N = N(u)$.

(b) ($\Rightarrow$) Let i be the inclusion map $N(v)\subset N(u)$. Since the composition $(\hat{u}^{-1})\,|_{N(u)}\circ i\circ \hat{v}\,|_{C\otimes B}$ is a C-comodule and B-module endomorphism of $C\otimes B$, by Lemma 1 there is $t\in \mathrm{Hom}(C,B)$ satisfying:

$$\begin{array}{ccc} C\otimes B & \overset{\hat{t}}{\longrightarrow} & C\otimes B \\ \wr\downarrow \hat{v} & \circlearrowleft & \wr\downarrow \hat{u} \\ N(v) & \hookrightarrow & N(u) \end{array}$$

Hence we have for $c\in C$

$$\sum_{(c)} c_{(1)} \otimes v(c_{(2)}) = \sum_{(c)} c_{(1)} \otimes u * t(c_{(2)}) \qquad \text{in } C \otimes A,$$

where we view $t \in \mathrm{Hom}(C, A)$ via $\mathrm{Hom}(C, B) \subset \mathrm{Hom}(C, A)$. Applying $\varepsilon \otimes \mathrm{id}_A$, we have $v = u * t$, so that $u^{-1} * v(C) \subset B$. ($\Leftarrow$) This is easily verified.

(c) This follows from (b). Q.E.D.

Let $\square_C$ be the cotensor product [D, §1], [T2, Appendix 2].

LEMMA 5. *Let $u \in \mathrm{Reg}(C, A)$ and let $D \subset C$ be a subcoalgebra. Then*

$$N(u\,|_D) = D \square_C N(u) = N(u) \cap D \otimes A.$$

PROOF. This follows by applying $D \square_C -$ to

$$C \otimes B \underset{\hat{u}}{\overset{\sim}{\rightarrow}} N(u) \quad (\subset C \otimes A).$$

In general, for a C-comodule V, $D \square_C V$ is a unique maximal D-subcomodule contained in V. In particular $D \square_C (C \otimes B) = D \otimes B$, so the first equality holds. The latter equality holds, since $D \square_C (C \otimes A) = D \otimes A$. Q.E.D.

For a C-comodule V, denote by V_0 the socle of V. In particular C_0 is the coradical of C, the direct sum of simple subcoalgebras $\subset C$, and $V_0 = C_0 \square_C V$.

PROPOSITION 6. *Let B be an algebra and let N be a B-module which has such a C-comodule structure $N \to C \otimes N$ that is a B-module map. (Then N_0 is a B-submodule of N.) Suppose that N is an injective C-comodule and that there is a C_0-comodule and B-module isomorphism $N_0 \simeq C_0 \otimes B$. Then there is a C-comodule and B-module isomorphism $N \simeq C \otimes B$.*

PROOF. Call the isomorphism $g : C_0 \otimes B \xrightarrow{\sim} N_0$. Since N is C-injective, the composition

$$C_0 \subset C_0 \otimes B \overset{g}{\underset{\sim}{\rightarrow}} N_0 \subset N$$

can be extended to a C-comodule map $f : C \to N$. Then

$$\bar{f} : C \otimes B \to N, \quad \bar{f}(c \otimes b) = f(c)b$$

is a C-comodule and B-module map which is an extension of g. Since $\bar{f}$ is injective on the socle $(C \otimes B)_0 = C_0 \otimes B$, $\bar{f}$ is injective. Since the "free" comodule $C \otimes B$ is C-injective [D, Cor. 1, p. 33] and since $\bar{f}(C_0 \otimes B) = N_0$, $\bar{f}$

is an isomorphism. Q.E.D.

In the following we use the next:

NOTATION 7. Let R be another algebra and fix an algebra map $\alpha: R \to A$. Define

$$B = \{a \in A \mid a\alpha(x) = \alpha(x)a, \ \forall x \in R\}.$$

Recall the definition of inner measurings in Introduction.

PROPOSITION 8. (a) *Let* $u, v \in \mathrm{Reg}(C, A)$. *We have*

$$\mathrm{inn}\, u = \mathrm{inn}\, v \Leftrightarrow u^{-1} * v \in \mathrm{Reg}(C, B) \Leftrightarrow u^{-1} * v \in \mathrm{Hom}(C, B)$$

$$\Leftrightarrow N(u) = N(v) \qquad \Leftrightarrow N(u) \supset N(v).$$

Thus $\mathrm{inn}\, u \leftrightarrow N(u)$ *gives rise to a* 1–1 *correspondence between the set of inner measurings* $C \to \mathrm{Hom}(R, A)$ *and the set of cleft B-forms of* $C \otimes A$.

(b) *Let* $D \subset C$ *be a subcoalgebra. Let* $\phi: C \to \mathrm{Hom}(R, A)$, $\psi: D \to \mathrm{Hom}(R, A)$ *be inner measurings and let* $N \subset C \otimes A$, $L \subset D \otimes A$ *be the corresponding cleft B-forms. Then*

$$\psi = \phi|_D \Leftrightarrow L = D \square_C N \Leftrightarrow L \subset N.$$

PROOF. (a) The first and second "$\Leftrightarrow$" are proved in the same way as [BCM, Lemma 1.13(1), p. 676]: One can write $\mathrm{inn}\, u = u * x\varepsilon * u^{-1}$, where $x\varepsilon(c) = \alpha(x)\varepsilon(c)$ for $c \in C$. Then one can show $\mathrm{inn}\, u = \mathrm{inn}\, v$ if and only if $u^{-1} * v(C) \subset B$ (and $v^{-1} * u(C) \subset B$). The remainder follows from Lemma 4.

(b) The first "$\Leftrightarrow$" follows from Lemma 5. The latter "$\Leftrightarrow$" follows from part (a) above, since $D \square_C N$ is a cleft B-form of $D \otimes A$. Q.E.D.

THEOREM 9. *For any measuring* $\phi: C \to \mathrm{Hom}(R, A)$, *there is a unique maximal subcoalgebra D of C such that* $\phi|_D$ *is inner.*

The 1–1 correspondence in Proposition 8 enables us to prove Theorem 9 in terms of cleft B-forms. To begin with, we show the existence of a maximal subcoalgebra whose action is inner. For this purpose it suffices by the Zorn lemma to prove:

LEMMA 10. *Let* $\{C_\lambda\}_{\lambda \in \Lambda}$ *be a set of subcoalgebras of C which is totally ordered under inclusion. Let* $N_\lambda \subset C_\lambda \otimes A$ $(\lambda \in \Lambda)$ *be cleft B-forms such that* $C_\lambda \subset C_\mu$ *implies* $N_\lambda \subset N_\mu$. *Write*

$$D = \bigcup_{\lambda} C_\lambda, \qquad N = \bigcup_{\lambda} N_\lambda.$$

Then N is a cleft B-form of $D \otimes A$.

PROOF. Applying $\varinjlim$ to the canonical isomorphisms

$$N_\lambda \otimes_B A \simeq C_\lambda \otimes A,$$

we have that N is a B-form of $D \otimes A$. To show N is cleft, by Proposition 6 we have only to prove Claims 11–12 below.

CLAIM 11. *N is an injective D-comodule.*

Since $C_\lambda \subset C_\mu$ implies $N_\lambda = C_\lambda \square_{C_\mu} N_\mu = C_\lambda \square_D N_\mu$ by Proposition 8(b), one has

$$N_\lambda = C_\lambda \square_D N.$$

To show any D-comodule map $h : V \to N$ can be extended to a D-comodule W including V, we may assume that V and W are finite dimensional [T2, p. 1527, ll.19–21]. Then there is C_λ such that W (hence V) is a C_λ-comodule (i.e., $W = C_\lambda \square_D W$). Since $h(V) \subset N_\lambda = C_\lambda \square_D N$ and since N_λ ($\simeq C_\lambda \otimes B$) is C_λ-injective, h can be extended to W. Thus N is D-injective.

CLAIM 12. *There is a D_0-comodule and B-module isomorphism $N_0 \simeq D_0 \otimes B$.*

Let E be a simple subcoalgebra $\subset D$. There is C_λ which includes E. Then we have

$$E \square_D N = E \square_{C_\lambda} C_\lambda \square_D N = E \square_{C_\lambda} N_\lambda \simeq E \otimes B.$$

Hence $N_0 \simeq D_0 \otimes B$, as is required. Q.E.D.

Let D, D' be two maximal subcoalgebras $\subset C$ whose actions are inner. Since $D_0 = (D' \cap D_0) \oplus E$ for some subcoalgebra $E \subset D_0$, we have $D' + D_0 = D' \oplus E$. By the maximality of D' we have $D_0 \subset D'$, so $D_0 \subset D'_0$. By symmetry we have $D_0 = D'_0$. Therefore the proof of Theorem 9 will be completed by the following lemma, which tells that "maximal" implies "unique maximal".

LEMMA 13. *Let $D, D' \subset C$ be subcoalgebras such that $D_0 = D'_0$. Let $N \subset D \otimes A$, $N' \subset D' \otimes A$ be cleft B-forms such that*

$$(D \cap D') \square_D N = (D \cap D') \square_{D'} N'.$$

Then $N + N'$ is a cleft B-form of $(D + D') \otimes A$. (So the inner measurings

$D \to \mathrm{Hom}(R, A)$ and $D' \to \mathrm{Hom}(R, A)$ corresponding to N and N' respectively can be extended uniquely to an inner measuring $D + D' \to \mathrm{Hom}(R, A)$.)

PROOF. Write $E = D \cap D'$, $L = E \square_D N = E \square_{D'} N'$. One has

$$E_0 = D_0 \cap D'_0 = D_0 = D'_0,$$

$$L_0 = E_0 \square_D N = D_0 \square_D N = N_0$$

$$= E_0 \square_{D'} N' = D'_0 \square_{D'} N' = N'_0.$$

We claim $L = N \cap N'$. Clearly $L \subset N \cap N'$. Since L ($\simeq E \otimes B$) is E-injective and since $N \cap N'$ ($\subset E \otimes A$) is an E-comodule such that $L_0 = (N \cap N')_0$, we have $L = N \cap N'$.

Let $g : E \otimes B \xrightarrow{\sim} L$ be an E-comodule and B-module isomorphism. Since $(D \otimes B)_0 = E_0 \otimes B$ and $N_0 = L_0$, it follows in the same way as the proof of Proposition 6 that g can be extended to a D-comodule and B-module isomorphism $f: D \otimes B \xrightarrow{\sim} N$. Similarly g is extended to $f' : D' \otimes B \xrightarrow{\sim} N'$. We have that there is a $(D + D')$-comodule and B-module isomorphism $(D + D') \otimes B \xrightarrow{\sim} N + N'$ from the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \to & (D \cap D') \otimes B & \to & (D \oplus D') \otimes B & \to & (D + D') \otimes B & \to 0 \\
 & \wr\downarrow g & & \wr\downarrow f \oplus f' & & & \\
0 \to L = & N \cap N' & \to & N \oplus N' & \to & N + N' & \to 0
\end{array}$$

The proof will be completed, if we show that $N + N'$ is a B-form of $(D + D') \otimes A$. This follows from the commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & (N \cap N') \otimes_B A & \to & (N \oplus N') \otimes_B A & \to & (N + N') \otimes_B A & \to 0 \\
 & \wr\downarrow & & \wr\downarrow & & \downarrow & \\
0 \to & (D \cap D') \otimes A & \to & (D \oplus D') \otimes A & \to & (D + D') \otimes A & \to 0
\end{array}$$

where all arrows are canonical ones. Q.E.D.

After the author submitted the earlier manuscript for publication, M. Takeuchi and M. Koppinen independently wrote to him a quick proof of Theorem 9. Here we sketch their proof. Let the notations be the same as in Notation 7 and Theorem 9. First, one can show from [T1, Lemma 14, p. 568] the following:

LEMMA A. *Let $D \subset C$ be a subcoalgebra. Then the restriction map* $\mathrm{Reg}(C, A) \to \mathrm{Reg}(D, A)$ *is a surjection.*

Let $\mathscr{X}$ be a set of all pairs (D, u) such that D is a subcoalgebra of C, $u \in \mathrm{Reg}(D, A)$ and $\phi|_D = \mathrm{inn}\, u$. Introduce into $\mathscr{X}$ a natural order determined by

$$(D, u) < (D', u') \Leftrightarrow D \subset D' \quad \text{and} \quad u = u'|_D.$$

By the Zorn Lemma, there is a maximal pair (D_m, u_m). The proof will be completed, if one shows that $D_m \supset D$ for any $(D, u) \in \mathscr{X}$. This follows, since one has the following:

LEMMA B. *Let (D, u), $(D', u') \in \mathscr{X}$. Then there is $v \in \mathrm{Reg}(D + D', A)$ such that $(D, u) < (D + D', v)$. Hence, if (D, u) is maximal, $D \supset D'$.*

PROOF. Write $E = D \cap D'$. By Proposition 8(a), $(u'|_E)^{-1} * (u|_E) \in \mathrm{Reg}(E, B)$. This can be extended to some $w \in \mathrm{Reg}(D', B)$ by Lemma A. Then $u|_E = (u' * w)|_E$ again by Proposition 8(a). Hence one can define a linear map $v : D + D' \to A$ by $v|_D = u$ and $v|_{D'} = u' * w$. This v is invertible by [T1, Lemma 14] and satisfies the required condition. Q.E.D.

Cleft forms remain useful for us to examine inner actions of coalgebras or Hopf algebras. See [Ma2, Section 3].

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REFERENCES

[BCM] R. J. Blattner, M. Cohen and S. Montgomery, *Crossed product and inner actions of Hopf algebras*, Trans. Am. Math. Soc. **298** (1986), 671–711.

[D] Y. Doi, *Homological coalgebra*, J. Math. Soc. Japan **33** (1981), 31–50.

[DT] Y. Doi and M. Takeuchi, *Hopf-Galois extensions of algebras, the Miyashita-Ulbrich action, and Azumaya algebras*, J. Algebra **121** (1989), 488–516.

[Ma1] A. Masuoka, *Coalgebra actions on Azumaya algebras*, Tsukuba J. Math. 107–112.

[Ma2] A. Masuoka, *On Hopf algebras with cocommutative coradicals*, J. Algebra, to appear.

[Mo] S. Montgomery, *Inner actions of Hopf algebras*, in *Ring Theory 1989*, Israel Mathematical Conference Proceedings, Vol. 1, Weizmann Science Press of Israel, Jerusalem, 1989, pp. 141–149.

[S] M. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.

[T1] M. Takeuchi, *Free Hopf algebras generated by coalgebras*, J. Math. Soc. Japan **23** (1971), 561–582.

[T2] M. Takeuchi, *Formal schemes over fields*, Comm. Algebra **5** (1977), 1483–1528.